Linear Algebra 1

08/10/2024

1 Systems of linear equations (5 + 1 + 5 + 2 + 2 + 5 + 5 = 25 pts)

Consider the cubic polynomial

$$p(x) = q + rx + sx^2 + tx^3$$

where $q, r, s, t \in \mathbb{F}$.

(a) Find a system of linear equations in the unknowns q, r, s, t such that

$$p(1) = p'(1) = 4, \quad p(2) = 14, \quad \text{and} \quad p'(2) = 17.$$

(b) Write down the corresponding augmented matrix.

(c) By performing elementary row operations, put the augmented matrix into row echelon form.

(d) Determine whether the system is consistent.

(e) Determine the *lead* and *free* variables.

(f) Put the augmented matrix into *reduced* row echelon form.

(g) Find the solution set.

REQUIRED KNOWLEDGE: **Gauss-Jordan elimination, row operations, reduced row echelon form, notions of lead/free variables.**

SOLUTION:

1a: Note that $p'(x) = r + 2sx + 3tx^2$ and

$$\begin{aligned} p(1) &= 4 = q + r + s + t \\ p'(1) &= 4 = r + 2s + 3t \\ p(2) &= 14 = q + 2r + 4s + 8t \\ p'(2) &= 17 = r + 4s + 12t. \end{aligned}$$

1b: Then, the augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 & 14 \\ 0 & 1 & 4 & 12 & 17 \end{bmatrix}$$

1c: To put the augmented matrix into row echelon form, we apply elementary row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 & 14 \\ 0 & 1 & 4 & 12 & 17 \end{bmatrix} \xrightarrow{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{1}} \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 7 & 10 \\ 0 & 1 & 4 & 12 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 7 & 10 \\ 0 & 1 & 4 & 12 & 17 \end{bmatrix} \xrightarrow[\textcircled{4} \leftarrow \textcircled{4} - 1 \cdot \textcircled{2}]{\textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 9 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 9 & 13 \end{bmatrix} \xrightarrow{\textcircled{4} \leftarrow \textcircled{4} - 2 \cdot \textcircled{3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The last matrix is in the row echelon form.

1d: From (1c), we see that the last column of the row echelon form does not have a leading 1, therefore the system is consistent.

1e: All unknowns are lead variables. There are no free variables.

1f: To obtain the row reduced echelon form, we continue applying elementary row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\textcircled{1} \leftarrow \textcircled{1} - 1 \cdot \textcircled{4}]{\substack{\textcircled{3} \leftarrow \textcircled{3} - 4 \cdot \textcircled{4} \\ \textcircled{2} \leftarrow \textcircled{2} - 3 \cdot \textcircled{4}}} \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{\textcircled{2} \leftarrow \textcircled{2} - 2 \cdot \textcircled{3} \\ \textcircled{1} \leftarrow \textcircled{1} - 1 \cdot \textcircled{3}}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} - 1 \cdot \textcircled{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

1g: The unique solution is $q = 4$, $r = -3$, $s = 2$, and $t = 1$, that is there is a unique polynomial

$$p(x) = 4 - 3x + 2x^2 + x^3$$

satisfying the conditions.

2 Partitioned matrices (10 + 10 = 20 pts)

Let A be $n \times n$ matrix. Let

$$M = \begin{bmatrix} I_n & A \\ A & I_n \end{bmatrix}.$$

(a) Show that M is nonsingular if and only if both $I_n - A$ and $I_n + A$ are nonsingular.

(b) Suppose that both $I_n - A$ and $I_n + A$ are nonsingular. Find the inverse of M.

REQUIRED KNOWLEDGE: **Partitioned matrices and nonsingularity.**

SOLUTION:

2a: For the 'if' part, suppose that both $I_n - A$ and $I_n + A$ are nonsingular. Let $\boldsymbol{x}, \boldsymbol{y}$ be n-vectors satisfying

$$\mathbf{0}_{2n} = M \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} I_n & A \\ A & I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}.$$

This leads to

$$\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0}_n \quad \text{and} \quad A\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}_n.$$

Then, we have $(I_n - A^2)\boldsymbol{y} = \mathbf{0}_n$. Since $(I_n - A^2) = (I_n - A)(I_n + A)$ and product of nonsingular matrices are nonsingular, we see that $(I_n - A^2)$ is nonsingular. Therefore, we obtain $\boldsymbol{y} = \mathbf{0}_n$. It, then, follows from $\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0}_n$ that $\boldsymbol{x} = \mathbf{0}_n$. Consequently, M is nonsingular.

For the 'only if' part, suppose that M is nonsingular. Let $\boldsymbol{x}$ be an n-vector such that $(I_n - A)\boldsymbol{x} = \mathbf{0}_n$. Note that

$$M \begin{bmatrix} \boldsymbol{x} \\ -\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} (I_n - A)\boldsymbol{x} \\ -(I_n - A)\boldsymbol{x} \end{bmatrix} = \mathbf{0}_{2n}.$$

Since M is nonsingular, we can conclude that $\boldsymbol{x}$ must be zero. Therefore, we see that $I_n - A$ is nonsingular. Now, let $\boldsymbol{x}$ be an n-vector such that $(I_n + A)\boldsymbol{x} = \mathbf{0}_n$. Note that

$$M \begin{bmatrix} \boldsymbol{x} \\ -\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} (I_n + A)\boldsymbol{x} \\ (I_n + A)\boldsymbol{x} \end{bmatrix} = \mathbf{0}_{2n}.$$

Since M is nonsingular, we can conclude that $\boldsymbol{x}$ must be zero. Therefore, we see that $I_n + A$ is nonsingular.

2b: From (2a), we know that M is nonsingular. Let U, V, W, X be $n \times n$ matrices and

$$\begin{bmatrix} U & V \\ W & X \end{bmatrix}$$

be the inverse of M. Then, we have

$$\begin{bmatrix} I_n & A \\ A & I_n \end{bmatrix} \begin{bmatrix} U & V \\ W & X \end{bmatrix} = I_{2n}.$$

In other words, we have

$$\begin{bmatrix} U + AW & V + AX \\ AU + W & AV + X \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

By solving V and W from the off-diagonal blocks, we obtain $V = -AX$ and $W = -AU$. From the diagonal blocks, we see that $(I_n - A^2)U = I_n$ and $(I_n - A^2)X = I_n$. Since $(I_n - A^2) =$

$(I_n - A)(I_n + A)$, $(I_n - A^2)$ is nonsingular. Let $B = (I_n - A^2)^{-1}$. Then, we have $U = X = B$ and $V = W = -AB$. As such,

$$M^{-1} = \begin{bmatrix} (I_n - A^2)^{-1} & -A(I_n - A^2)^{-1} \\ -A(I_n - A^2)^{-1} & (I_n - A^2)^{-1} \end{bmatrix}.$$

3 Eigenvalues and eigenvectors (2 + 15 + 3 = 20 pts)

Let $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times m}$.

(a) How many rows and columns do the matrices AB and BA have?

(b) Show that the nonzero eigenvalues of AB and BA are the same.

(c) Find A and B such that AB is nonsingular and BA is singular. (Take $m = 1$ and $n = 2$)

Required Knowledge: **Matrix multiplication, eigenvalues/eigenvectors, and (non)singularity.**

Solution:

3a: The matrix AB is $m \times m$ and BA is $n \times n$.

3b: Because of symmetry, it is enough to show that every nonzero eigenvalue of AB is also an eigenvalue of BA. Let $(\lambda, \boldsymbol{x})$ be an eigenpair of AB such that $\lambda \neq 0$. This means that

$$AB\boldsymbol{x} = \lambda\boldsymbol{x}. \tag{1}$$

Now, multiply both sides from left by B:

$$BAB\boldsymbol{x} = \lambda B\boldsymbol{x}.$$

This shows that $(BA)(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$. Since $\lambda \neq 0$ by assumption and $\boldsymbol{x}$ is a nonzero vector as an eigenvector, we see from (1) that $B\boldsymbol{x}$ must be nonzero. Therefore, λ is an eigenvalue of BA.

3c: Take

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and observe that

$$AB = 1 \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, AB is nonsingular. Observe that $\det(BA) = 0$ as BA has a zero row. So, it is singular.

4 Determinants and diagonalization $(3 + 5 + (2 + 15) = 25 \text{ pts})$

Let $M(n) \in \mathbb{R}^{n\times n}$ be given by

$$[M(n)]_{ij} = \begin{cases} a & \text{if } i = j \\ b & \text{if } i = j+1 \\ c & \text{if } i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$M(4) = \begin{bmatrix} a & c & 0 & 0 \\ b & a & c & 0 \\ 0 & b & a & c \\ 0 & 0 & b & a \end{bmatrix}.$$

(a) Compute the determinant of $M(n)$ for $n \in \{1, 2, 3\}$.

(b) Let $d(n) = \det M(n)$. Find real numbers x, y such that

$$d(n) = xd(n-1) + yd(n-2)$$

for all $n \geqslant 3$.

(c) Take $a = 5$, $b = c = 2$.

(i) Let $\boldsymbol{e}(n) = \begin{bmatrix} d(n-1) \\ d(n-2) \end{bmatrix}$. Find a matrix A such that

$$\boldsymbol{e}(n+1) = A\boldsymbol{e}(n)$$

for all $n \geqslant 3$.

(ii) Diagonalize A, compute A^n, and find $d(n)$.

Required Knowledge: **Nonsingularity and partitioned matrices.**

Solution:

4a: Note that

$$M(1) = a, \quad M(2) = \begin{bmatrix} a & c \\ b & a \end{bmatrix}, \quad \text{and} \quad M(3) = \begin{bmatrix} a & c & 0 \\ b & a & c \\ 0 & b & a \end{bmatrix}.$$

Then, we have

$$\begin{aligned} \det M(1) &= a \\ \det M(2) &= a^2 - bc \\ \det M(3) &= a \det\left(\begin{bmatrix} a & c \\ b & a \end{bmatrix}\right) - c \det\left(\begin{bmatrix} b & c \\ 0 & a \end{bmatrix}\right) \\ &= a(a^2 - bc) - abc = a^3 - 2abc. \end{aligned}$$

4b: Let $n \geqslant 3$. By cofactor expansion along row 1, we have that

$$d(n) = ad(n-1) - c \det\left(\begin{bmatrix} b & q \\ 0_{n-2,1} & M(n-2) \end{bmatrix}\right)$$

where q is an $n-1$-row vector. By cofactor expansion along column 1, wee see that

$$\det\left(\begin{bmatrix} b & q \\ 0_{n-2,1} & M(n-2)\end{bmatrix}\right) = bd(n-2).$$

Therefore, we obtain

$$d(n) = ad(n-1) - bcd(n-2)$$

for all $n \geqslant 3$. As such, $x = a$ and $y = -bc$.
4c(i): For $a = 5$ and $b = c = 2$. We have

$$d(n) = 5d(n-1) - 4d(n-2)$$

for $n \geqslant 3$. Therefore, we see that

$$\boldsymbol{e}(n+1) = \begin{bmatrix} d(n) \\ d(n-1)\end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 0\end{bmatrix}\begin{bmatrix} d(n-1) \\ d(n-2)\end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 1 & 0\end{bmatrix}\boldsymbol{e}(n).$$

Hence,

$$A = \begin{bmatrix} 5 & -4 \\ 1 & 0\end{bmatrix}.$$

4c(ii): To diagonalize A, we begin with finding its characteristic polynomial:

$$p_A(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 5 & 4 \\ -1 & \lambda\end{bmatrix}\right) = \lambda(\lambda-5)+4 = \lambda^2 - 5\lambda + 4.$$

Then, we see that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. Next, we find eigenvectors by noting that

$$\mathbf{0} = (\lambda_1 I - A)\boldsymbol{x} = \begin{bmatrix} -4 & 4 \\ -1 & 1\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} \iff x_1 = x_2$$

and that

$$\mathbf{0} = (\lambda_2 I - A)\boldsymbol{y} = \begin{bmatrix} -1 & 4 \\ -1 & 4\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \iff y_1 = 4y_2.$$

Therefore, we can choose $\begin{bmatrix} 1 \\ 1\end{bmatrix}$ as an eigenvector corresponding to λ_1 and $\begin{bmatrix} 4 \\ 1\end{bmatrix}$ corresponding to λ_2. Now, let

$$X = \begin{bmatrix} 1 & 4 \\ 1 & 1\end{bmatrix}$$

and note that

$$AX = \begin{bmatrix} 5 & -4 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 1\end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4\end{bmatrix} = XD$$

where D is a diagonal matrix. Since

$$X^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -4 \\ -1 & 1\end{bmatrix},$$

we see that

$$A^n = XD^nX^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^n\end{bmatrix}\begin{bmatrix} 1 & -4 \\ -1 & 1\end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & 4^{n+1} \\ 1 & 4^n\end{bmatrix}\begin{bmatrix} 1 & -4 \\ -1 & 1\end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1-4^{n+1} & 4^{n+1}-4 \\ 1-4^n & 4^n-4\end{bmatrix}.$$

Now, observe that

$$d(n) = \begin{bmatrix} 1 & 0\end{bmatrix}\boldsymbol{e}(n+1) = \begin{bmatrix} 1 & 0\end{bmatrix}A^{n-2}\boldsymbol{e}(3)$$

and

$$\boldsymbol{e}(3) = \begin{bmatrix} d(2) \\ d(1)\end{bmatrix} = \begin{bmatrix} a^2 - bc \\ a\end{bmatrix} = \begin{bmatrix} 21 \\ 5\end{bmatrix}.$$

Therefore, we see that

$$d(n) = -\frac{1}{3}\begin{bmatrix} 1 & 0\end{bmatrix}\begin{bmatrix} 1-4^{n-1} & 4^{n-1}-4 \\ 1-4^{n-2} & 4^{n-2}-4\end{bmatrix}\begin{bmatrix} 21 \\ 5\end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1-4^{n-1} & 4^{n-1}-4\end{bmatrix}\begin{bmatrix} 21 \\ 5\end{bmatrix} = \frac{4^{n+1}-1}{3}.$$